Int. J. Solids Structures Vol. 26, No. 2, pp. 149–163, 1990
Printed in Great Britain.

0020–7683/90 $3.00 + .00

THE PLASTIC SPIN IN VISCOPLASTICITY

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(*Received* 14 *September* 1988; *in revised form* 4 *May* 1989)

Abstract—The importance of the role of plastic spin in the rate-dependent response of materials at large deformations is the main objective of this work. After a brief presentation of a general constitutive framework for viscoplasticity at large strains, an isotropic/kinematic hardening and an orthotropic viscoplastic model are used to analyze the stress–strain response under simple shear and biaxial loading at different rates. A clear understanding of the role of plastic spin is achieved by obtaining closed-form analytical expressions for different stress values, in which the plastic spin constitutive parameters interrelate with the strain rate and other more conventional model constants. Such analytical expressions allow for a direct evaluation of the capabilities of the models to account for large deformations and rate dependence as exhibited by available experimental data, and provide guidance towards the proper choice of constitutive parameters.

1. INTRODUCTION

In large deformation constitutive modeling, one of the important objectives is the proper description of the evolution of the material substructure. The substructure can macroscopically be described by tensorial structure (or internal) variables, which also provide the anisotropic properties via their orientation (Onat, 1982). The study of the substructural evolution becomes, thus, tantamount to the study of evolution of the structure variables. In rate theories the evolution is analytically expressed by rate constitutive equations.

The proper rate definition in these equations has been, and still is, debated. Mandel (1971) proposed a macroscopic theory in which the rate is defined as corotational in reference to a substructural spin ω, which is different from the material spin $\mathbf{W}$ of the continuum. In fact, Mandel (1971) and Kratochvil (1971) proposed kinematics in which $\omega = \mathbf{W} - \mathbf{W}^p$, where the spin $\mathbf{W}^p$ is the antisymmetric part of the plastic velocity gradient and is given by proper constitutive relations. Such constitutive relations for $\mathbf{W}^p$ were first presented by Dafalias (1983a, b, 1984a, 1985) and Loret (1983). Dafalias (1984a, 1985) named $\mathbf{W}^p$ the plastic spin, a terminology which has been adopted by other researchers.

The role of the plastic spin is clearly associated with the definition of ω, as explained above. Therefore, it is equally important for both rate-independent and rate-dependent plasticity (or viscoplasticity), as in fact shown by the original work of Mandel (1971) who addressed both cases within the same general framework. While Mandel kept a general perspective mainly due to the fact that he had not proposed any definite form for $\mathbf{W}^p$, Dafalias (1983a, b, 1984a, 1985) and Loret (1983) showed by concrete examples the role of plastic spin in rate-independent plasticity. Other workers in the field also contributed to this task directly or indirectly, among them Lee *et al.* (1983), Fressengeas and Molinari (1983), Paulun and Pecherski (1985, 1987) and Im and Atluri (1987). The role of plastic spin in viscoplasticity was investigated in some detail by Dafalias (1984b), Dafalias and Aifantis (1984), Anand (1985), Bammann and Aifantis (1987), and Dafalias and Rashid (1989).

The purpose of this paper is to present firstly a straightforward and simple general formulation of viscoplasticity at large strains, where the role of plastic spin is clear and unambiguous in relation to the basic equations of kinematics and kinetics. Secondly, two viscoplastic constitutive models are presented within the framework of the plastic spin concept, one with kinematic hardening and one for orthotropic symmetries. And thirdly, concrete examples for simple shear and biaxial loading are analyzed using both models. What is perhaps most interesting and different from other works is the clear understanding of the interrelation between the role of plastic spin and the rate dependence which characterizes viscoplasticity. This is achieved by obtaining analytical expressions in closed form for different values of stresses, in which the parameters defining the plastic spin interrelate with the strain rate and other more conventional material constants. The analytical solutions give an explicit insight into the inherent capabilities of different models to account for rate dependence, hence, they provide proper guidance for future use in relation to modeling actual experimental data.

2. KINEMATICS AND KINETICS IN VISCOPLASTICITY AT LARGE DEFORMATIONS

The material state variables can be defined at the current configuration κ in terms of the Cauchy stress $\boldsymbol{\sigma}$, temperature θ, and a set of structure variables (or internal variables) consisting of second-order tensors $\mathbf{a}$ and scalars k ($\mathbf{a}$ and k may imply many entities). Tensors of other orders can also be included, but are omitted for simplicity.

The basic kinematical assumption is the well-known multiplicative decomposition of the deformation gradient into elastic and plastic (or inelastic) parts (Lee, 1969), supplemented by the notion of a material substructure defined by $\mathbf{a}$ and k which obeys different (although linked) kinematics than the continuum (Mandel, 1971). Bypassing the details of the kinematical analysis and assuming small elastic strains for simplicity [such details and the effect of large elastic strains are thoroughly discussed in Dafalias (1985, 1987, 1988)], one can express the important result of the Eulerian kinematics by

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \tag{1}$$

$$\mathbf{W} = \boldsymbol{\omega} + \mathbf{W}^p \tag{2}$$

where $\mathbf{D}$ is the rate of deformation tensor (symmetric part of the velocity gradient), $\mathbf{D}^e$ and $\mathbf{D}^p$ the elastic and plastic parts of it. As already mentioned in the Introduction, $\mathbf{W}$ is the material spin tensor (antisymmetric part of the velocity gradient), $\boldsymbol{\omega}$ is the rigid body spin of the substructure and $\mathbf{W}^p$ the plastic spin. It follows from eqn (2) that $\mathbf{W}^p$ expresses the rate of rotation of the continuum with respect to its substructure in the process of inelastic deformations. Mandel (1971) identified $\boldsymbol{\omega}$ as the spin of a triad of director vectors attached to the substructure. Dafalias (1983b, 1987) has shown that one can dispense, in general, with the sometimes elusive notion of director vectors, but still keep the concept of $\boldsymbol{\omega}$ and $\mathbf{W}^p$ and the validity of eqn (2).

While eqns (1) and (2) express the kinematics for any kind of inelastic material response, the kinetics specify the constitutive equations pertinent to a certain class of behavior. The corotational rate of a tensor $\mathbf{a}$ with respect to $\boldsymbol{\omega}$ is defined by

$$\mathring{\mathbf{a}} = \dot{\mathbf{a}} - \boldsymbol{\omega}\mathbf{a} + \mathbf{a}\boldsymbol{\omega} \tag{3}$$

and let : denote the trace operation over two adjacent pairs of indices. Then, the following set of constitutive equations describe fully the elastothermoviscoplastic material response

$$\mathbf{D}^e = \mathscr{L}^{-1} : \mathring{\boldsymbol{\sigma}} + \mathbf{T}\dot{\theta} \tag{4}$$

$$\mathbf{D}^p = \Phi\mathbf{N}^p, \quad \mathbf{W}^p = \Phi\boldsymbol{\Omega}^p \tag{5}$$

$$\mathring{\mathbf{a}} = \Phi_i\bar{\mathbf{a}}_i, \quad \dot{k} = \Phi_i\bar{k}_i. \tag{6}$$

The $\mathscr{L}$ represent the elastic moduli, $\mathbf{T}$ is the thermal expansion tensor, Φ and Φ_i are non-negative scalar valued overstress functions and $\mathbf{N}^p$, $\boldsymbol{\Omega}^p$, $\bar{\mathbf{a}}_i$ and $\bar{k}_i$ define the "direction" of $\mathbf{D}^p$, $\mathbf{W}^p$, $\mathring{\mathbf{a}}$ and $\dot{k}$, respectively. The summation convention over i is implied in eqns $(6)_1$ and $(6)_2$. An important property of Φ and Φ_i is that they may be zero for certain range of values of $\boldsymbol{\sigma}$ and θ, hence, arresting further deformation and the evolution of the internal substructure. It is possible to have $\Phi = 0$ while some $\Phi_i \neq 0$, implying $\mathbf{D}^p = 0$ and $\mathbf{W}^p = \mathbf{0}$ while $\mathring{\mathbf{a}} \neq \mathbf{0}$ and $\dot{k} \neq 0$, a phenomenon called "hesitation" by Mandel (1971). Notice the common Φ for $\mathbf{D}^p$ and $\mathbf{W}^p$ and the multiple Φ_i for $\mathring{\mathbf{a}}$ and $\dot{k}$. Invariance requirements under superposed rigid body rotation render all tensor and scalar valued constitutive functions of eqns (4)–(6) isotropic functions of their variables $\boldsymbol{\sigma}$, θ, $\mathbf{a}$, and k. A detailed account of such invariance and its consequence is given in Dafalias (1987, 1988).

Equations (4) and $(6)_1$ express the fact that corotational rates of $\boldsymbol{\sigma}$ and $\mathbf{a}$ must be in reference to $\boldsymbol{\omega}$ and not $\mathbf{W}$, as commonly practiced, since $\boldsymbol{\sigma}$ and $\mathbf{a}$ are attached to the substructure which spins by $\boldsymbol{\omega}$ (Mandel, 1971). In fact, substituting $\mathbf{W}-\mathbf{W}^p$ for $\boldsymbol{\omega}$ according to eqn (2) and using eqns (1)–(6), one can rewrite eqns (4) and $(6)_1$ as

$$\overset{\nabla}{\boldsymbol{\sigma}} = \mathscr{L}:[\mathbf{D}-\mathbf{T}\dot{\theta}]-\Phi(\mathscr{L}:\mathbf{N}^p+\boldsymbol{\Omega}^p\boldsymbol{\sigma}-\boldsymbol{\sigma}\boldsymbol{\Omega}^p) \tag{7}$$

$$\overset{\nabla}{\mathbf{a}} = \Phi_i\bar{\mathbf{a}}_i-\Phi(\boldsymbol{\Omega}^p\mathbf{a}-\mathbf{a}\boldsymbol{\Omega}^p) \tag{8}$$

where a superposed ∇ implies the corotational rate with respect to $\mathbf{W}$ (classical Jaumann rate).

With a proper norm denoted by $|\ \ |$, eqn $(5)_1$ yields

$$\Phi(\boldsymbol{\sigma},\theta,\mathbf{a},k) = |\mathbf{D}^p|/|\mathbf{N}^p|. \tag{9}$$

For a given form of Φ, eqn (9) can be considered the equation of the so-called "dynamic" yield surface in viscoplasticity. Often, $|\mathbf{D}^p|$ can be considered a measure of the total rate of deformation if one neglects the elastic rate of deformation, hence the "size" of the dynamic yield surface is rendered rate-dependent according to eqn (9).

In reference to eqn $(6)_1$, it is very common that one of the $\bar{\mathbf{a}}_i$ is proportional to $\mathbf{N}^p$, i.e. $\bar{\mathbf{a}}_i = H\mathbf{N}^p$ with the corresponding $\Phi_i = \Phi$. Also, another $\bar{\mathbf{a}}_{i-1} = c\bar{\mathbf{a}}$ with $\Phi_{i-1} = \Phi$. Hence, based on eqn (9) and with $C = c/|\mathbf{N}^p|$, eqn $(6)_1$ can be written as

$$\overset{\circ}{\mathbf{a}} = H\mathbf{D}^p+C|\mathbf{D}^p|\bar{\mathbf{a}}+\Phi_{i-2}\bar{\mathbf{a}}_{i-2}. \tag{10}$$

According to eqn (10), $\overset{\circ}{\mathbf{a}}$ is given by three distinct terms. The "hardening" term $H\mathbf{D}^p$ and the "dynamic recovery" term $C|\mathbf{D}^p|\bar{\mathbf{a}}$ occur only when $\mathbf{D}^p \neq \mathbf{0}$ (i.e. $\Phi > 0$), while the "static recovery" terms $\Phi_{i-2}\bar{\mathbf{a}}_{i-2}$ operate always as long as $\Phi_{i-2} > 0$. Without the existence of the static recovery the $\mathbf{a}$ would depend on the plastic strain path ("orbit" traced by $\mathbf{D}^p$) but not the plastic strain rate (speed at which the "orbit" is being traversed). This occurs approximately at very high strain rates (very large $|\mathbf{D}^p|$), where the static recovery has no time to influence $\overset{\circ}{\mathbf{a}}$. Even in this case, however, the $\boldsymbol{\sigma}$ depends on $|\mathbf{D}^p|$ as seen from eqn (9). A similar equation to (10) can also be written for $\dot{k}$.

3. SPECIFIC THERMOVISCOPLASTIC MODELS

A key function in thermoviscoplasticity is the overstress function Φ. Since the original work by Perzyna (1963) many forms of Φ have been presented, and the reader is referred to recent comprehensive papers by Chaboche and Rousselier (1983) and Krempl (1987). In most cases the way to define Φ goes as follows. A positive scalar-valued isotropic function J of the state variables having the dimension of stress is defined. The form of J is motivated by the expression of a classical yield surface in rate-independent plasticity. The difference $J-k$ from a reference stress variable k determines the so-called overstress measure. The Φ is a function of $J-k$ such that when $J-k \leqslant 0 \Rightarrow \Phi = 0$, with $\Phi > 0$ otherwise. It is possible that $k = 0$, in which case $\Phi > 0$ always since $J > 0$. In addition, the $\mathbf{N}^p$ is defined by $\mathbf{N}^p = \partial J/\partial\boldsymbol{\sigma}$ in what can be called an associated flow rule in viscoplasticity. The $\boldsymbol{\Omega}^p$ for the plastic spin in eqn $(5)_2$ is defined according to the works of Dafalias (1983a, b, 1984a, 1985) and Loret (1983).

In the following such models will be presented along the lines of the foregoing discussion, properly adjusted to account for the role of plastic spin at large deformations.

3.1. *Isotropic and kinematic hardening model*

Motivated by the original work of Armstrong and Frederick (1966), Chaboche (1977) presented a viscoplastic constitutive model with a power law overstress function which, supplemented here by an equation for the plastic spin, can be expressed within the framework of eqns (5) and (6) by

$$\mathbf{D}^p = \frac{3}{2}\left\langle\frac{J(\mathbf{s}-\boldsymbol{\alpha})-(k_0+R)}{V}\right\rangle^n \frac{\mathbf{s}-\boldsymbol{\alpha}}{J(\mathbf{s}-\boldsymbol{\alpha})} \tag{11}$$

$$\mathbf{W}^p = \frac{3}{2}\left\langle\frac{J(\mathbf{s}-\boldsymbol{\alpha})-(k_0+R)}{V}\right\rangle^n \frac{\eta(\boldsymbol{\alpha}\mathbf{s}-\mathbf{s}\boldsymbol{\alpha})}{J(\mathbf{s}-\boldsymbol{\alpha})} = \tfrac{1}{2}\rho(\boldsymbol{\alpha}\mathbf{D}^p-\mathbf{D}^p\boldsymbol{\alpha}) \tag{12}$$

$$\overset{o}{\boldsymbol{\alpha}} = \tfrac{2}{3}h_\alpha \mathbf{D}^p - c_r\dot{\bar{\varepsilon}}^p\boldsymbol{\alpha} - c_s J^{m-1}(\boldsymbol{\alpha})\boldsymbol{\alpha} \tag{13}$$

$$\dot{R} = (H_\alpha - C_r R)\dot{\bar{\varepsilon}}^p - C_s R^m \tag{14}$$

with s being the deviatoric part of $\boldsymbol{\sigma}$, $\langle\ \rangle$ the Macauley brackets and the definitions

$$J(\mathbf{a}) = (\tfrac{3}{2}\mathbf{a}:\mathbf{a})^{1/2} \tag{15}$$

$$\dot{\bar{\varepsilon}}^p = |\mathbf{D}^p| = (\tfrac{2}{3}\mathbf{D}^p:\mathbf{D}^p)^{1/2}. \tag{16}$$

The $\bar{\varepsilon}^p$ can be recognized as the cumulative equivalent plastic strain obtained by integration of eqn (16). The $\boldsymbol{\alpha}$ represents the deviatoric "back-stress" tensor, while $k = k_0+R$ is the reference stress whose variable part R represents the isotropic hardening (initially $k = k_0$). Equations (13) and (14) are of the form of eqn (10) with clearly defined hardening, dynamic and static recovery terms. One can also recognize that $\mathbf{N}^p = \partial J(\mathbf{s}-\boldsymbol{\alpha})/\partial\boldsymbol{\sigma} = (3/2)(\mathbf{s}-\boldsymbol{\alpha})/J$, hence, $\Phi = \langle(J-k)/V\rangle^n$. All the scalar valued parameters k_0, V, n, ρ, h_α, c_r, c_s, H_α, C_r, C_s and m are positive isotropic functions of the state variables $\boldsymbol{\sigma}$, θ, $\boldsymbol{\alpha}$, R and $\bar{\varepsilon}^p$.

With reference to the key eqns $(5)_2$ and (12) for $\mathbf{W}^p$ and with the above definition of Φ it follows that $\boldsymbol{\Omega}^p = (3/2)\eta(\boldsymbol{\alpha}\mathbf{s}-\mathbf{s}\boldsymbol{\alpha})/J$, as originally suggested by Dafalias (1983a) and Loret (1983) based on the first generator $\boldsymbol{\alpha}\mathbf{s}-\mathbf{s}\boldsymbol{\alpha}$ for the representation of antisymmetric tensor-valued isotropic functions of s and $\boldsymbol{\alpha}$. The transition from the first to the second expression for $\mathbf{W}^p$ in eqn (12) is achieved by observing that $\boldsymbol{\alpha}\mathbf{s}-\mathbf{s}\boldsymbol{\alpha} = \boldsymbol{\alpha}(\mathbf{s}-\boldsymbol{\alpha})-(\mathbf{s}-\boldsymbol{\alpha})\boldsymbol{\alpha}$, using eqn (11) to express $\mathbf{s}-\boldsymbol{\alpha}$ in terms of $\mathbf{D}^p$ and setting $\rho = 2\eta$. The expression for $\mathbf{W}^p$ in terms of $\mathbf{D}^p$ was first proposed in eqn (23) of Dafalias (1983a) and today constitutes a basic equation in many works dealing with kinematic hardening models at large deformations. Based on eqns (11), (15) and (16), the equation corresponding to (9) of the general development becomes

$$J(\mathbf{s}-\boldsymbol{\alpha}) = [\tfrac{3}{2}(\mathbf{s}-\boldsymbol{\alpha}):(\mathbf{s}-\boldsymbol{\alpha})]^{1/2} = k_0+R+V(\dot{\bar{\varepsilon}}^p)^{1/n}. \tag{17}$$

Equation (17) is the dynamic yield surface with the rate dependence appearing explicitly via the term $V(\dot{\bar{\varepsilon}}^p)^{1/n}$, and implicitly via the values of $\boldsymbol{\alpha}$ and R which depend on the rate of deformation when the role of the static recovery terms is significant, as discussed after eqn (10). To illustrate this rate effect one can integrate eqn (14) for two values of m in closed form, assuming constant coefficients H_α, C_r, C_s, a constant rate $\dot{\bar{\varepsilon}}^p$ and $R = 0$ at $\bar{\varepsilon}^p = 0$. For $m = 1$, eqn (14) becomes a classical linear differential equation, while for $m = 2$ it becomes a Riccati equation. With the algebra of integration omitted, the results are given by

$$R = R_\infty[1-F(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p)]. \tag{18}$$

For $m = 1$

$$R_\infty = H_\alpha \Big/ \left(C_r + \frac{C_s}{\dot{\bar{\varepsilon}}^p}\right) \tag{19a}$$

$$F = \exp\left[-\left(C_r + \frac{C_s}{\dot{\bar{\varepsilon}}^p}\right)\bar{\varepsilon}^p\right]. \tag{19b}$$

For $m = 2$

$$R_\infty = \left[\left(\frac{C_r}{C_s}\right)^2\left(\frac{\dot{\bar{\varepsilon}}^p}{2}\right)^2 + \frac{H_\alpha}{C_s}\dot{\bar{\varepsilon}}^p\right]^{1/2} - \left(\frac{C_r}{C_s}\right)\frac{\dot{\bar{\varepsilon}}^p}{2} \tag{20a}$$

$$F = \left[\left(2 + \frac{C_r}{C_s}\frac{\dot{\bar{\varepsilon}}^p}{R_\infty}\right)^{-1} - \left[\left(2 + \frac{C_r}{C_s}\frac{\dot{\bar{\varepsilon}}^p}{R_\infty}\right)^{-1} - 1\right]\exp\left\{\left(C_r + 2R_\infty\frac{C_r}{\dot{\bar{\varepsilon}}^p}\right)\bar{\varepsilon}^p\right\}\right]^{-1}. \tag{20b}$$

Observe that as $\bar{\varepsilon}^p \to \infty$, $F \to 0$ and $R \to R_\infty$. The importance of the static recovery term is determined by the value of $C_s/\dot{\bar{\varepsilon}}^p$ compared to C_r. For very slow rates, i.e. $\dot{\bar{\varepsilon}}^p \to 0$, $C_s/\dot{\bar{\varepsilon}}^p \to \infty$ and it follows from eqns (18)–(20) that $R \to 0$ for all $\bar{\varepsilon}^p$. On the other hand, for very high rates, i.e. $\dot{\bar{\varepsilon}}^p \to \infty$, $C_s/\dot{\bar{\varepsilon}}^p \to 0$ and it follows from eqn (14) that for all values of m the R is independent of $\dot{\bar{\varepsilon}}^p$ and given by

$$R = (H_\alpha/C_r)[1 - \exp(-C_r\bar{\varepsilon}^p)]. \tag{21}$$

For the uniaxial tension/compression case under stress σ, eqns (11), (13), (14) and (17) become, respectively,

$$\dot{\varepsilon}^p = \operatorname{sgn}(\sigma - a)\left\langle\frac{|\sigma - a| - (k_0 + R)}{V}\right\rangle^n \tag{22}$$

$$\dot{a} = (h_\alpha - \operatorname{sgn}(\dot{\varepsilon}^p)c_r a)\dot{\varepsilon}^p - c_s|a|^{m-1}a \tag{23}$$

$$\dot{R} = (H_\alpha - C_r R)|\dot{\varepsilon}^p| - C_s R^m \tag{24}$$

$$|\sigma - a| = k_0 + R + V|\dot{\varepsilon}^p|^{1/n} \tag{25}$$

where use of $J(\mathbf{s} - \boldsymbol{\alpha}) = (3/2)|s_{11} - \alpha_{11}| = |\sigma - a|$, $J(\boldsymbol{\alpha}) = (3/2)|\alpha_{11}| = |a|$, and $\dot{\bar{\varepsilon}}^p = |D_{11}| = |\dot{\varepsilon}^p|$ according to eqns (15) and (16) was made, together with the obvious definitions $a = (3/2)\alpha_{11}$, ε^p = the logarithmic uniaxial plastic strain and sgn (*) = sign of *.

Equation (25) is useful for fitting uniaxial experimental data with the effect of a, R and $\dot{\varepsilon}^p$ clearly shown. The R is obtained by eqns (18)–(20) with $|\varepsilon^p|$ and $|\dot{\varepsilon}^p|$ substituting for $\bar{\varepsilon}^p$ and $\dot{\bar{\varepsilon}}^p$, and the a can be obtained also in closed form by the integration of eqn (23); in fact, the result of such integration can be obtained by substituting in eqns (18)–(20) the $|a|$ for R (and of course $|a|_\infty$ for R_∞), h_α for H_α, c_r for C_r, c_s for C_s and again $|\varepsilon^p|$, $|\dot{\varepsilon}^p|$ for $\bar{\varepsilon}^p$, $\dot{\bar{\varepsilon}}^p$ for a monotonic change from $a = 0$. All observations made after eqn (20) pertaining to the limits of R as $\bar{\varepsilon}^p$ and $\dot{\bar{\varepsilon}}^p$ tend to 0 and ∞, apply for $|a|$ as well.

Often it is desirable to distribute the kinematic and isotropic hardening according to a weighting factor $0 \leqslant \xi \leqslant 1$, with $\xi = 0$ and $\xi = 1$ representing purely kinematic and purely isotropic hardening, respectively. This means that $a = (1-\xi)(a \pm R)$ and $R = \xi|a \pm R|$ (the $\pm$ for $\dot{\varepsilon}^p \gtrless 0$), which implies that the foregoing distribution is achieved by substituting $(1-\xi)h_\alpha$, $(1-\xi)^{1-m}c_s$, ξh_α, $\xi^{1-m}c_s$ and c_r for h_α, c_s, H_α, C_s and C_r, respectively, in eqns (13), (14), (24) and (25) [the c_r remains as in eqns (13) and (23)]. For this case it can be shown that eqns (19) and (20) yield $|a \pm R|$ in lieu of R if one substitutes h_α, c_r and c_s for H_α, C_r and C_s, respectively.

3.2. Orthotropic and transversely isotropic model

Consider an orthotropic material with $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ unit vectors along the axes of orthotropy $\hat{x}_i$, $i = 1, 2, 3$. If $\mathbf{a}_1 = \mathbf{n}_1 \otimes \mathbf{n}_1$ and $\mathbf{a}_2 = \mathbf{n}_2 \otimes \mathbf{n}_2$, it can be shown (Liu, 1982) that the state variables are $\boldsymbol{\sigma}$, θ, $\mathbf{a}_1$ and $\mathbf{a}_2$. Hence, motivated by the orthotropic yield criterion proposd by Hill (1950), and denoting by a superposed ^ the tensor components in reference to the orthotropic axes, one can define the following J

$$J(\boldsymbol{\sigma}, \mathbf{a}_1, \mathbf{a}_2) = [A(\hat{\sigma}_{11} - \hat{\sigma}_{22})^2 + B(\hat{\sigma}_{22} - \hat{\sigma}_{33})^2 + C(\hat{\sigma}_{33} - \hat{\sigma}_{11})^2$$
$$+ 2D\hat{\sigma}_{23}^2 + 2E\hat{\sigma}_{31}^2 + 2F\hat{\sigma}_{12}^2]^{1/2} = [(A + B + 4C - 2E)\,\mathrm{tr}^2(\mathbf{a}_1\mathbf{s}) + (A + C$$
$$+ 4B - 2D)\,\mathrm{tr}^2(\mathbf{a}_2\mathbf{s}) + 2(-A + 2B + 2C - D - E + F)\,\mathrm{tr}(\mathbf{a}_1\mathbf{s})\,\mathrm{tr}(\mathbf{a}_2\mathbf{s})$$
$$+ 2(F - D)\,\mathrm{tr}(\mathbf{a}_1\mathbf{s}^2) + 2(F - E)\,\mathrm{tr}(\mathbf{a}_2\mathbf{s}^2) + (E + D - F)\,\mathrm{tr}\,\mathbf{s}^2]^{1/2} \tag{26}$$

where the second expression for J in terms of $\mathbf{s}$ refers to any set of axes since it depends on invariants (Dafalias and Rashid, 1989). Observe then that for the isotropic case $A = B = C = 1$, $D = E = F = 3$, and eqn (26) yields $J = \sqrt{3}(\mathrm{tr}\,\mathbf{s}^2)^{1/2}$.

Based on eqn (26), the orthotropic thermoviscoplastic flow rule is given by

$$\mathbf{D}^p = \left\langle \frac{J(\boldsymbol{\sigma}, \mathbf{a}_1, \mathbf{a}_2) - \sqrt{2}(k_0 + R)}{V} \right\rangle^n \frac{\partial J}{\partial \boldsymbol{\sigma}} \tag{27}$$

with $\dot{R}$ given by the evolution eqn (14), hence, R given by eqns (18)–(20) for $m = 1$ and 2. On the basis of eqn (16), the dynamic yield surface is obtained from eqns (26) and (27) as

$$J(\boldsymbol{\sigma}, \mathbf{a}_1, \mathbf{a}_2) = \sqrt{2}(k_0 + R) + V\left[\left(\tfrac{2}{3}\mathrm{tr}\left(\frac{\partial J}{\partial \boldsymbol{\sigma}}\right)^2\right)^{-1/2} \dot{\bar{\varepsilon}}^p\right]^{1/n}. \tag{28}$$

The factor $\sqrt{2}$ was placed in front of $k_0 + R$ in eqns (27) and (28) in order to identify the latter as the static yield stress (i.e. when $\dot{\bar{\varepsilon}}^p \simeq 0$) in uniaxial tension/compression when eqn (26) degenerates to the isotropic case.

In line with the procedure in Dafalias (1983b, 1984a, 1985) and Loret (1983) for rate-independent plasticity where the pertinent algebra can be found, the plastic spin $\mathbf{W}^p$ is obtained according to eqn (5) by multiplying a linear combination of the generators $\mathbf{a}_1\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{a}_1$, $\mathbf{a}_2\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{a}_2$ and $\mathbf{a}_1\boldsymbol{\sigma}\mathbf{a}_2 - \mathbf{a}_2\boldsymbol{\sigma}\mathbf{a}_1$ giving $\boldsymbol{\Omega}^p$, with the overstress function of eqn (27) yielding

$$\hat{W}^p_{12} = \hat{\eta}_3\hat{D}^p_{12}, \quad \hat{W}^p_{13} = \hat{\eta}_2\hat{D}^p_{13}, \quad \hat{W}^p_{23} = \hat{\eta}_1\hat{D}^p_{23} \tag{29}$$

for the components in reference to the orthotropic axes, with the $\hat{\eta}_i$s functions of the state variables.

Since the structure variables $\mathbf{a}_i$ ($i = 1, 2$) specify only the orientation of the orthotropic axes, they are purely orientational and their evolution is given according to eqns $(6)_1$ or (8) by (Dafalias, 1985)

$$\overset{\circ}{\mathbf{a}}_i = \mathbf{0} \quad \text{or} \quad \overset{\nabla}{\mathbf{a}}_i = \mathbf{a}_i\mathbf{W}^p - \mathbf{W}^p\mathbf{a}_i. \tag{30}$$

Hence, eqns (29) are useful in determining the evolution of $\mathbf{a}_i$ according to eqns (30), as it will be illustrated by examples in the sequel.

A thermoviscoplastic model with transversely isotropic symmetries of the fifth class characterized by rotational symmetry around $\mathbf{n}_1$ can be obtained from the orthotropic model by simply setting $A = C$, $E = F$, $D = A + 2B$ in eqn (26) and $\hat{\eta}_1 = 0$, $\hat{\eta}_2 = \hat{\eta}_3 = \eta$ in eqn (29). The interpretation of the values of η in relation to fiber-reinforced or layered media can be found in Dafalias (1984a, 1985) and Dafalias and Rashid (1989).

4. SIMPLE SHEAR

The first example illustrating the effect of plastic spin in viscoplasticity is that of simple shear defined by the velocity gradient components

$$D_{12} = D_{21} = W_{12} = -W_{21} = \tfrac{1}{2}\dot{\gamma}, \quad D_{ij} = W_{ij} = 0 \quad \text{for other } i,j. \tag{31}$$

In what follows the material response will be considered rigid-plastic for simplicity so that $\mathbf{D}^p = \mathbf{D}$. The analysis will be performed using the two viscoplastic models presented in the previous section. Corresponding analyses have been presented already for rate-independent plasticity by Dafalias (1985), and pertinent results will be used here in order to avoid repeating algebraic manipulations, with emphasis placed on the additional effect of rate dependence.

4.1. *Use of the isotropic/kinematic hardening model*

With the simplifying assumptions $\sigma_{33} = 0$, $\alpha_{13} = \alpha_{23} = 0$ and $\alpha_{11}+\alpha_{22} = 0$, eqns (2), (3), (11)–(13), (15)–(17) and (31) yield

$$\sigma_{11} = \alpha_{11} = -\sigma_{22} = -\alpha_{22}, \quad \sigma_{13} = \sigma_{23} = 0 \tag{32a}$$

$$\sigma_{12} = \alpha_{12} + \operatorname{sgn}(\dot{\gamma})\frac{1}{\sqrt{3}}\left[k_0 + R + V\left(\frac{|\dot{\gamma}|}{\sqrt{3}}\right)^{1/n}\right] \tag{32b}$$

while the evolution of α_{11} and α_{12} with γ is governed by the system

$$\frac{d\alpha_{11}}{d\gamma} = -\operatorname{sgn}(\dot{\gamma})\frac{1}{\sqrt{3}}\left[c_r + \frac{\sqrt{3}c_s}{|\dot{\gamma}|}[3(\alpha_{11}^2+\alpha_{12}^2)]^{(m-1)/2}\right]\alpha_{11} + (1-\rho\alpha_{11})\alpha_{12} \tag{33a}$$

$$\frac{d\alpha_{12}}{d\gamma} = -\operatorname{sgn}(\dot{\gamma})\frac{1}{\sqrt{3}}\left[c_r + \frac{\sqrt{3}c_s}{|\dot{\gamma}|}[3(\alpha_{11}^2+\alpha_{12}^2)]^{(m-1)/2}\right]\alpha_{12} - (1-\rho\alpha_{11})\alpha_{11} + \tfrac{1}{3}h_\alpha. \tag{33b}$$

In the process of deriving eqns (32b) and (33) the relations $\dot{\bar{\varepsilon}} = |\dot{\gamma}|/\sqrt{3}$ and $J(\boldsymbol{\alpha}) = [3(\alpha_{11}^2+\alpha_{12}^2)]^{1/2}$ were used. Equations (33) are derived from eqn (13) on the basis of eqn (3) and the computation of $\boldsymbol{\omega} = \mathbf{W}-\mathbf{W}^p$, with $\mathbf{W}^p$ given by eqn (12). Such computation yields $\omega_{12} = W_{12} - W_{12}^p = (\dot{\gamma}/2)(1-\rho\alpha_{11})$, clearly indicating the effect of the plastic spin in eqn (33) via the component $W_{12}^p = (\dot{\gamma}/2)\rho\alpha_{11}$.

Observe that the isotropic hardening affects only the value of σ_{12} in eqn (32b) via R, the latter being computed from eqns (18)–(20) with the aforementioned value of $\dot{\bar{\varepsilon}}^p$ and $\bar{\varepsilon}^p = |\gamma|/\sqrt{3}$ for monotonic change of γ. The rate of deformation affects implicitly the values of σ_{11}, σ_{22}, σ_{12} in eqn (32) via α_{11}, α_{22}, α_{12} and R, and explicitly only the value of σ_{12} in eqn (32b) via the last term premultiplied by V (which can be considered the most important rate effect). Finally, the temperature effect is incorporated in the dependence of all relevant parameters on θ. The foregoing observations are very important for numerical simulations of actual data, because they show what can and what cannot be done before even such simulations are attempted. For example, for sufficiently high rates the term $c_s/|\dot{\gamma}|$ tends to zero, hence, the $\boldsymbol{\alpha}$ and R do not depend on the rate; consequently one expects to find only an explicit σ_{12} rate-dependence according to eqn (32b), while the values of σ_{11}, σ_{22} from eqn (32a) are not rate sensitive. For experimental data which show the opposite, i.e. a σ_{11}, σ_{22} rate sensitivity as in Montheillet *et al.* (1984), it may not be possible to use the foregoing model for their simulation in light of the previous observations.

In order to obtain an insight of the behavior of the highly non-linear system of the differential eqns (33) by analytical means, the particular assumption $m = 1$ will be adopted, henceforth. Then, it is seen that eqn (33) can be derived from eqns (37) of Dafalias (1985) by substituting in the latter the $c_r + (\sqrt{3}c_s/|\dot{\gamma}|)$ for c_r. With this substitution all conclusions derived from eqn (37) in Dafalias (1985) can now be repeated for eqn (33), supplemented by some new observations and omitting the algebra which can be found in the above reference.

For $\rho = 0$, the solution of eqn (33) for monotonic γ can be obtained in closed-form as

$$\alpha_{11} = \frac{h_\alpha}{3+[c_r+(\sqrt{3}c_s/|\dot{\gamma}|)]^2}\left[1-\exp\left[-\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)|\gamma|\right]\left(\cos\gamma\right.\right.$$
$$\left.\left.+\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)\sin|\gamma|\right)\right]+\exp\left[-\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)|\gamma|\right](\alpha^0_{11}\cos\gamma+\alpha^0_{12}\sin\gamma) \quad (34a)$$

$$\alpha_{12} = \frac{(\mathrm{sgn}\,\gamma)h_\alpha}{3+[c_r+(\sqrt{3}c_s/|\dot{\gamma}|)]^2}\left[\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}+\exp\left[-\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)|\gamma|\right]\left(\sin|\gamma|\right.\right.$$
$$\left.\left.-\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)\cos\gamma\right)\right]-\exp\left[-\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)|\gamma|\right](\alpha^0_{11}\sin\gamma-\alpha^0_{12}\cos\gamma) \quad (34b)$$

with α^0_{11}, α^0_{12} the values of α_{11}, α_{12} at $\gamma = 0$. The oscillatory nature of eqn (34) transfers to the values of σ_{11} and σ_{12} according to eqn (32). From eqn (34) it can be seen that as $|\dot{\gamma}| \to \infty$ the rate-effect on the α_{11}, α_{12} disappears, while for $|\dot{\gamma}| \to 0$ the $\alpha_{11} \to 0$ and $\alpha_{12} \to 0$.

For $\rho \neq 0$, one can first conclude that $\rho > 0$ (Dafalias, 1985). Then, a unique equilibrium point α^e_{11}, α^e_{12} is given by

$$\alpha^e_{11} = (1/3\rho)[2+(3\sqrt{q}-p)^{1/3}-(3\sqrt{q}+p)^{1/3}] \quad (35a)$$

$$\alpha^e_{12} = (\mathrm{sgn}\,\gamma)\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)\alpha^e_{11}\Big/(1-\rho\alpha^e_{11}) \quad (35b)$$

with

$$p = 3\left[\left(c_r+\frac{\sqrt{3}c_s}{|\dot{\gamma}|}\right)^2-\tfrac{1}{2}\rho h_\alpha+\tfrac{1}{3}\right] \quad (36a)$$

$$q = \frac{1}{9}\left[p^2+\left(\left(c_r+\frac{\sqrt{3}c_s}{|\dot{\gamma}|}\right)^2+\rho h_\alpha-1\right)^3\right] \quad (36b)$$

under the condition $q > 0$ which is sufficient and necessary for α^e_{11} to be the unique real root of the cubic equation

$$\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)^2\alpha^e_{11}\Big/(1-\rho\alpha^e_{11}) = \rho(\alpha^e_{11})^2-\alpha^e_{11}+\tfrac{1}{3}h_\alpha. \quad (37)$$

It can be shown from the signs of the two members of eqn (37) (Dafalias, 1985), that

$$0 < \alpha^e_{11} < 1/\rho. \quad (38)$$

Hence, $\alpha^e_{11} \to 0$ as $\rho \to \infty$ and consequently $\rho(\alpha^e_{11})^2 \to 0$ as $\rho \to \infty$. Observing from eqns (35b) and (37) that α^e_{12} can be expressed by the right-hand side of eqn (37) premultiplied by $(\mathrm{sgn}\,\gamma)((c_r/\sqrt{3})+c_s/|\dot{\gamma}|)^{-1}$, and using eqn (34) when $\rho = 0$, one can state that

For $\rho = 0$

$$\alpha^e_{11} = \frac{(h_\alpha/3)}{1+\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)^2}, \quad \alpha^e_{12} = \frac{(\operatorname{sgn}\dot{\gamma})\frac{h_\alpha}{3}\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)}{1+\left(\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}\right)^2}. \tag{39a}$$

For $\rho \to \infty$

$$\alpha^e_{11} \to 0, \quad \alpha^e_{12} \to \frac{(\operatorname{sgn}\dot{\gamma})(h_\alpha/3)}{\frac{c_r}{\sqrt{3}}+\frac{c_s}{|\dot{\gamma}|}}. \tag{39b}$$

Equations (39) are useful for curve fitting experimental data by varying ρ, since they explicitly show the corresponding possible range of values for α_{11} and α_{12}.

Along the lines of the procedure in Dafalias (1985), an analysis based on Liapunov's second method for stability shows that asymptotic convergence of eqn (33) in the whole towards α^e_{11}, α^e_{12} is obtained under the sufficient conditions

$$c_r+\frac{\sqrt{3}c_s}{|\dot{\gamma}|} > \frac{1}{3}, \quad \left(c_r+\frac{\sqrt{3}c_s}{|\dot{\gamma}|}\right)^2+\rho h_\alpha > 1. \tag{40}$$

Furthermore, with Δ the discriminant of the characteristic equation of the linear approximation of eqn (33) given by

$$\Delta = \frac{1}{3}\left(c_r+\frac{\sqrt{3}c_s}{|\dot{\gamma}|}\right)^2 \rho^2(\alpha^e_{11})^2+4(1-\rho\alpha^e_{11})^3(2\rho\alpha^e_{11}-1) \tag{41}$$

and because the phase portraits of eqn (33) and its linear approximation are the same, one has that the equilibrium point α^e_{11}, α^e_{12} is a stable node when $\Delta \geqslant 0$, and a stable spiral when $\Delta < 0$. In the latter case oscillation of the stress can be observed. Notice that since $|\dot{\gamma}|$ can affect the sign of Δ, oscillations may or may not occur depending on the rate.

Illustration of the foregoing is presented by the plots of the σ_{12} and $\sigma_{22}(=-\sigma_{11})$, normalized by k_0, versus γ in Fig. 1 for $\rho = 0.3$ and $\rho = 2$, where ρ is normalized by k_0^{-1}. The remaining constants have the values $h_\alpha = 3$, $c_r = \sqrt{3}$, $c_s = 0.02\ \mathrm{s}^{-1}$, $n = 5$, $V = 1$, while $R = 0$ (no isotropic hardening). The values of h_α and V are normalized by k_0. The results for a high rate $\dot{\gamma} = 0.5\sqrt{3}\ \mathrm{s}^{-1}$ and a low rate $\dot{\gamma} = 0.007\sqrt{3}\ \mathrm{s}^{-1}$ are shown with corresponding graphs. In order to evaluate the effect of the recovery term via c_s, the same problem was analyzed for $c_s = 0$. For the high rate the graphs were indistinguishable, since $c_s = 0$ is practically equivalent to $c_s/|\dot{\gamma}|$ when $|\dot{\gamma}|$ is high, as follows from the system eqn (33). For the low rate, the $c_s = 0$ yields the $\sigma_{12}-\gamma$ plot shown by the discontinuous line, above the corresponding plot for $c_s = 0.02$; the difference shows the effect that $c_s = 0$ has on σ_{12} via α_{12} according to eqns (33) and (32b). The σ_{22} for $c_s = 0$ is identical to σ_{22} for the high rate, according to the previous observation. The effect of ρ is clearly shown to be an increase of σ_{12} and decrease of σ_{22} (absolutely) as ρ increases. The limits of eqn (39) apply at the extreme cases $\rho = 0$ and $\rho \to \infty$.

4.2. *Use of the orthotropic and transversely isotropic model*

Assume that the orthotropic axes $\hat{x}_1$, $\hat{x}_2$ form an angle ϕ with x_1, x_2 (positive counterclockwise from x_1 to $\hat{x}_1$) and $\hat{x}_3 = x_3$. Then, according to the steps in Dafalias (1985) for rate-independent plasticity and the use of eqn (27), the non-zero stress components are given by

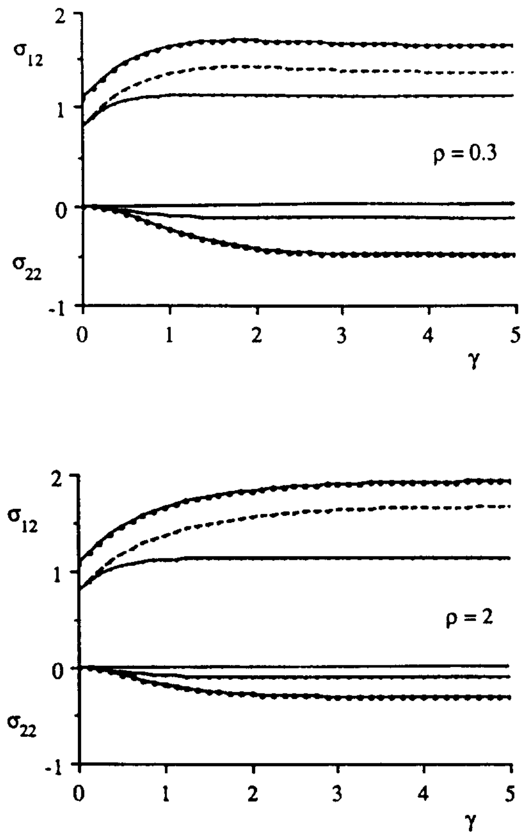

Fig. 1. The stress–strain response in simple shear with the kinematic hardening model for two values of ρ and two rates. Lines with thick and thin dots correspond to high and low rates, respectively. The discontinuous lines correspond to the case $c_s = 0$ and the low rate. The stress is normalized by k_0.

$$\frac{\hat{\sigma}_{11}}{B} = -\frac{\hat{\sigma}_{22}}{C} = \frac{\hat{\sigma}_{12}}{(X/F\tan 2\phi)} = \frac{J}{R}\,\mathrm{sgn}\,(\gamma)\,\mathrm{sgn}\,(\sin 2\phi) \tag{42}$$

with $X = AB+BC+CA$ and $R = [X(B+C+(2X/F\tan^2 2\phi))]^{1/2}$. The evolution of ϕ is specified by the differential equation [based on eqns (29), (30)]

$$\frac{\mathrm{d}\phi}{\mathrm{d}\gamma} = \tfrac{1}{2}(\eta\cos 2\phi - 1) \tag{43}$$

with η the plastic spin coefficient function of $\boldsymbol{\sigma}$, $\mathbf{a}_1$, $\mathbf{a}_2$ and θ. Observe that the rate dependence and isotropic hardening appear via J in eqn (42), as given by eqn (28) with $\dot{\bar{\varepsilon}}^p = |\dot{\gamma}|/\sqrt{3}$.

An exhaustive study of eqns (42) and (43) for rate-independent response is presented in Dafalias and Rashid (1989) based on the original work by Dafalias (1984a). This study can be extended here with J substituting for what appeared as k in the aforementioned reference. For example, for cubic orthotropic symmetries where $A = B = C = \alpha$ and $F = \beta$, use of the stress transformation from $\hat{x}_i$ to x_i in eqn (42) yields

$$-\frac{\sigma_{11}}{J} = \frac{\sigma_{22}}{J} = \mathrm{sgn}\,(\dot{\gamma}\cos 2\phi)\left(\frac{3\alpha}{\beta}-1\right)\frac{\sin 2\phi}{\left[6\alpha\left(\frac{3\alpha}{\beta}+\tan^2 2\phi\right)\right]^{1/2}} \tag{44a}$$

$$\frac{\sigma_{12}}{J} = \mathrm{sgn}\,(\dot{\gamma})\left[\frac{1}{6\alpha}\left(1+\left(\frac{3\alpha}{\beta}-1\right)\cos^2 2\phi\right)\right]^{1/2} \tag{44b}$$

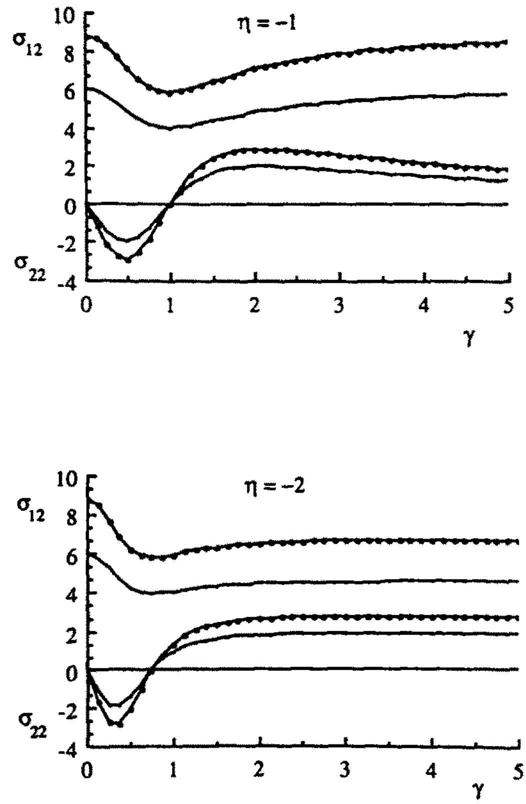

Fig. 2. The stress–strain response in simple shear with the orthotropic model for two values of η and two rates. Lines with thick and thin dots correspond to high and low rates, respectively. The initial value of ϕ is $\phi_0 = 0$. The stress is normalized by k_0.

while eqn (43) can be integrated in closed-form yielding for different values of η the ϕ as a function γ. An interesting point of eqn (44) is that the rate dependence through J affects not only σ_{12}, but also σ_{11} and σ_{22}, contrary to the case with kinematic hardening, eqn (32). In addition it can be shown that the σ_{11}, σ_{22} can change sign in the process of shearing, depending on the value of η.

An illustration of the foregoing is presented in Fig. 2 for $\eta = -1$ and $\eta = -2$. The other parameters attain the values $\alpha = 1/2$, $\beta = 2/3$, $n = 5$, $V[(2/3)\,\mathrm{tr}\,(\partial J/\partial\sigma)^2]^{-1/2n} = 10$ with V normalized by k_0 and $\phi_0 = 0$ (initial value of ϕ), while invariably $R = 0$. Given the foregoing values of η and ϕ_0 one has from eqn (43) (Dafalias and Rashid, 1989) that $\tan\phi = -\gamma$ for $\eta = -1$, and $\tan\phi = -\sqrt{3}(1-\exp(-\sqrt{3}\gamma))/(1+\exp(-\sqrt{3}\gamma))$ for $\eta = -2$. The plots of σ_{12} and σ_{22}, normalized by k_0, versus γ are shown for two rates $\dot{\gamma} = 0.5\sqrt{3}\,\mathrm{s}^{-1}$ and $\dot{\gamma} = 0.05\sqrt{3}\,\mathrm{s}^{-1}$. The rate increase results into an increase of the absolute value of the stresses, but the points of maximum, minimum or zero occur at the same γ. Particularly interesting is the change of sign of σ_{22} from negative (compressive) to positive (tensile), at the same γ for the different $\dot{\gamma}$. Such a change depends on the value of η which controls the evolution of ϕ. The response shown in Fig. 2 is qualitatively similar to some experimental data presented in Montheillet *et al.* (1984).

5. BIAXIAL PLANE STRESS

The second example, under the assumption of $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ and rigid-plastic response, can be defined by the velocity gradient components

$$D_{22} = rD_{11}, \quad D_{33} = -(D_{11}+D_{22}), \quad D_{ij} = 0 \quad \text{for} \quad i \neq j, \quad W_{ij} = 0. \tag{45}$$

The analysis is presented in terms of the kinematic hardening and the orthotropic models.

5.1. *Use of the isotropic/kinematic hardening model*

Equations (2), (3), (11)–(13) and (15)–(17) in conjunction with eqn (45) yield

$$\sigma_{11} = 2\alpha_{11}+\alpha_{22}+(\text{sgn } D_{11})\frac{2+r}{[3(1+r+r^2)]^{1/2}}J(\mathbf{s}-\boldsymbol{\alpha}) \tag{46a}$$

$$\sigma_{22} = \alpha_{11}+2\alpha_{22}+(\text{sgn } D_{11})\frac{1+2r}{[3(1+r+r^2)]^{1/2}}J(\mathbf{s}-\boldsymbol{\alpha}) \tag{46b}$$

$$\sigma_{12} = \alpha_{12} \tag{46c}$$

where $J(\mathbf{s}-\boldsymbol{\alpha})$ is given by eqn (17) with $\dot{\bar{\varepsilon}}^p = |D_{11}|\sqrt{2}(1+r+r^2)^{1/2}$; in addition, with $\alpha_+ = (\alpha_{11}+\alpha_{22})/2$, $\alpha_- = (\alpha_{11}-\alpha_{22})/2$ and ε the logarithmic strain along x_1, they also yield

$$\mathrm{d}\alpha_+/\mathrm{d}\varepsilon = (1+r)(h_\alpha/3)-(\text{sgn } D_{11})c\alpha_+ \tag{47a}$$

$$\mathrm{d}\alpha_-/\mathrm{d}\varepsilon = (1-r)(h_\alpha/3)-(\text{sgn } D_{11})c\alpha_- +\rho(1-r)\alpha_{12}^2 \tag{47b}$$

$$\mathrm{d}\alpha_{12}/\mathrm{d}\varepsilon = -[(\text{sgn } D_{11})c+\rho(1-r)\alpha_-]\alpha_{12} \tag{47c}$$

$$c = (2c_r/\sqrt{3})(1+r+r^2)^{1/2}+(c_s/|\dot{\varepsilon}|)J^{m-1}(\boldsymbol{\alpha}). \tag{47d}$$

The rate dependence of the stress appears directly via $J(\mathbf{s}-\mathbf{a})$ in eqn (46) as given by eqn (17), and indirectly via $c_s/|\dot{\varepsilon}|$ in eqn (47d) which affects α_{ij}.

In the following the assumption $m = 1$ is made. Then, eqn (47a) yields, for monotonic change of ε under constant $\dot{\varepsilon}$

$$\alpha_+ = \alpha_+^0 \exp(-c|\varepsilon|)+(\text{sgn } \varepsilon)\frac{(1+r)h_\alpha}{3c}[1-\exp(-c|\varepsilon|)]. \tag{48}$$

For $\alpha_{12}^0 = 0$ and/or $r = 1$, and/or $\rho = 0$ (the last case implies zero plastic spin), eqns (47b) and (47c) are uncoupled, can be integrated and in combination with eqn (48) yield

$$\alpha_{11} = \alpha_{11}^0 \exp(-c|\varepsilon|)+(\text{sgn } \varepsilon)\frac{2h_\alpha}{3c}[1-\exp(-c|\varepsilon|)] \tag{49a}$$

$$\alpha_{22} = \alpha_{22}^0 \exp(-c|\varepsilon|)+(\text{sgn } \varepsilon)\frac{2rh_\alpha}{3c}[1-\exp(-c|\varepsilon|)] \tag{49b}$$

$$\alpha_{12} = \alpha_{12}^0 \exp(-c|\varepsilon|). \tag{49c}$$

When $\alpha_{12}^0 \neq 0$, $\rho \neq 0$ and $r \neq 1$, by setting $\mathrm{d}\alpha_-/\mathrm{d}\varepsilon = 0$ and $\mathrm{d}\alpha_{12}/\mathrm{d}\varepsilon = 0$ in eqns (47b) and (47c) in order to find the equilibrium values α_{11}^e, α_{22}^e and α_{12}^e, one has in combination with eqn (48)

$$\alpha_{11}^e = (\text{sgn } \varepsilon)\frac{2h_\alpha}{3c},\quad \alpha_{22}^e = (\text{sgn } \varepsilon)\frac{2rh_\alpha}{3c},\quad \alpha_{12}^e = 0, \tag{50}$$

same as the ones obtained for $|\varepsilon| \to \infty$ from eqn (49). With the origin transferred at α_{ij}^e, eqns (47b) and (47c) become

$$\mathrm{d}\bar{\alpha}_-/\mathrm{d}\varepsilon = -(\text{sgn } \varepsilon)c\bar{\alpha}_- +\rho(1-r)\bar{\alpha}_{12}^2 \tag{51a}$$

$$\mathrm{d}\bar{\alpha}_{12}/\mathrm{d}\varepsilon = -(\text{sgn } \varepsilon)[c+(\rho(1-r)^2h_\alpha/3c)]\bar{\alpha}_{12}-\rho(1-r)\bar{\alpha}_-\bar{\alpha}_{12} \tag{51b}$$

where $\bar{\alpha}_- = \alpha_- -\alpha_-^e$ and $\bar{\alpha}_{12} = \alpha_{12}-\alpha_{12}^e$. Considering the Liapunov function $U = (1/2)(\bar{\alpha}_-^2 + \bar{\alpha}_{12}^2)$ one can easily compute using eqn (51) that $\mathrm{d}U/\mathrm{d}|\varepsilon| = -[c(\bar{\alpha}_-^2 +\bar{\alpha}_{12}^2)+(\rho(1-r)^2h_\alpha/3c)\bar{\alpha}_{12}^2]$, which is negative. Hence, the system of eqns (49b) and (49c) converges towards the equilibrium point eqn (50) in the whole space (i.e. irrespective of initial values). Furthermore, the characteristic equation of the linear approximation of eqn (51) (i.e. without the $\bar{\alpha}_{12}^2$ and $\bar{\alpha}_{-}\bar{\alpha}_{12}$ terms) has the real negative eigenvalues $v_1 = -c$ and $v_2 = -(c+(\rho(1-r)^2 h_\alpha/3c))$, thus the equilibrium point eqn (50) is a stable node.

5.2. *Use of the orthotropic model*

With $\hat{x}_1$, $\hat{x}_2$ forming an angle ϕ with x_1, x_2 and $\hat{x}_3 = x_3$, a similar procedure to the rate-independent results of Dafalias (1985) yields

$$\frac{\hat{\sigma}_{11}}{Q_1} = \frac{\hat{\sigma}_{22}}{Q_2} = \frac{\hat{\sigma}_{12}}{Q_3} = \frac{J}{Q}(\text{sgn } D_{11}) \tag{52}$$

where J is given by eqn (28) with $\dot{\bar{\varepsilon}}^p = |D_{11}|\sqrt{2}(1+r+r^2)^{1/2}$. The Q_1, Q_2, Q_3 and Q are functions of A, B, C, F, ϕ and r, as shown explicitly after eqn (46) in Dafalias (1985), where also it was obtained

$$\tan\phi = \tan\phi_0 \exp[(r-1)\eta\varepsilon]. \tag{53}$$

The new piece of information here is the rate dependence of the stress via J in eqn (52).

6. DISCUSSION AND CONCLUSION

The framework of viscoplasticity presented in the previous sections contained mainly one novel aspect with respect to classical formulations: the constitutive equations of the plastic spin. For the particular models examined, this focuses on the parameters ρ for kinematic hardening, and η for orthotropic symmetries. In general, ρ and η may depend on the state variables in a way which can be determined by purely macroscopic observations. For example, it was shown in Montheillet *et al.* (1984) that increasing temperature decreases the absolute value of σ_{22} in fixed-end torsion experiments. According to the previous analysis this suggests an increasing ρ with θ, if the kinematic hardening model is to be employed. Paulun and Pecherski (1985, 1987), using geometrical arguments, proposed a dependence of ρ on $\bar{\varepsilon}^p$.

Along a different line of thought Dafalias and Aifantis (1984) used the single slip kinetics and kinematics and a scale-invariance argument (Aifantis, 1984, 1987) to conclude that if Q is the multiplier of α in the evolution equation for $\overset{\circ}{\alpha}$, then $(\dot{\rho}/\rho) = -Q$. In reference to eqn (13), the foregoing relation yields

$$\rho = \rho_0 \exp\left[c_r\bar{\varepsilon}^p + c_s\int_0^t J^{m-1}(\alpha)\,dt\right]. \tag{54}$$

This line of approach was applied by Bammann and Aifantis (1987) and Zbib and Aifantis (1988) to corresponding models. Notice from eqn (54) that if $m = 1$, $\rho = \rho_0 \exp[c_r\bar{\varepsilon}^p + c_s t]$, which yields the unreasonable result that even without plastic deformation the ρ increases with t towards infinity. Hence, the $m = 1$ must be excluded if one follows an approach based on eqn (54).

In this paper the analysis was performed with constant values of ρ and η, rendering it possible to obtain certain analytical expressions (limits, condition for convergence, etc.) in closed form. Such examples with constant ρ and η are representative of what would be the response for variable ρ and η, due to the global character of the convergence criteria, and in fact can be used to properly define such variations. For example, motivated by some intermediate steps in the analysis of the work by Dafalias and Aifantis (1984), one can propose $\rho = \rho_s - (\rho_s - \rho_0)\exp(c^*\bar{\varepsilon}^p)$, i.e. an exponential variation of ρ with $\bar{\varepsilon}^p$ from ρ_0 to

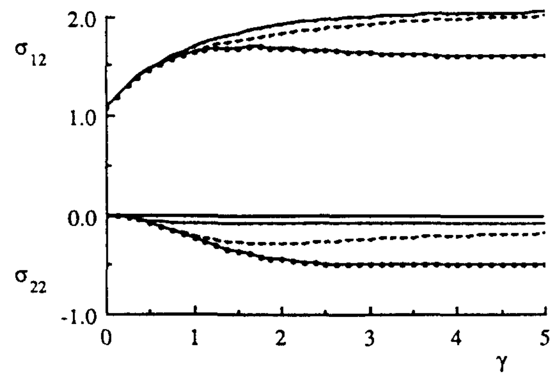

Fig. 3. The stress–strain response in simple shear with the kinematic hardening model for $\rho = 0.1$ (lines with thick dots), $\rho = 10$ (lines with thin dots) and an exponentially varying ρ with γ from 0.1 to 10 (discontinuous lines). The stress is normalized by k_0.

ρ_s. Figure 3 illustrates the behavior in simple shear for such a ρ variation with $\rho_0 = 0.1$, $\rho_s = 10$ and $c^* = -0.2$. The other constants assume the same values used for the graphs in Fig. 1 for $\dot{\gamma} = 0.5\sqrt{3}\ s^{-1}$. The corresponding graph in Fig. 3 (discontinuous line) lies between the graphs obtained for constant $\rho = 0.1$ and $\rho = 10$, converging asymptotically towards the latter. Observe the progressive decrease of $|\sigma_{22}|$ with γ after an initial increase, which is to be expected since $|\sigma_{22}|$ is higher for $\rho = 0.1$ than $\rho = 10$. Observations of this sort can be very useful when one attempts to fit experimental data showing similar trends.

In conclusion, the role of the plastic spin in viscoplasticity is similar to that in rate-independent plasticity, as already known in principle by the work of Mandel (1971). In this paper a concrete step-by-step presentation is made on the use of the plastic spin in both a general viscoplasticity theory and some particular cases of constitutive models and loading conditions. The goal was to obtain closed-form analytical rather than numerical results in an attempt to deepen the understanding of the constitutive response with plastic spin, in order to be used for actual rate-dependent modeling.

Acknowledgements—The support by the NSF Grant No. MSM-8619229 is gratefully acknowledged. Mr. Hanwook Cho, graduate student at U.C. Davis, provided the plots of the figures.

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